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ON THE USE OF EULER'S THEOREM ON ROTATIONS FOR THE SYNTHESIS OF ATTITUDE CONTROL SYSTEMS

by George Meyer

Ames Research Center

Moffett Field, Calif.

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SUMMARY

The problem of controlling the rotational position (attitude) of a rigid body in three dimensions is discussed. Several control laws are synthesized for this six-dimensional nonlinear control problem by means of some well-known techniques of classical mechanics.

The system input, output, and error are represented by 3×3 orthogonal matrices. Euler's theorem on rotations is employed to express the error matrix in terms of the angle of rotation and the direction cosines of the real eigenvector of the error matrix. It is noted that the angle of rotation is a convenient scalar representation of the system error. A class of control laws for which the control torque is a function of the real eigenvector of the error matrix and the angular velocity of the controlled body is synthesized. Conditions are stated for which the system governed by such control laws is asymptotically stable everywhere.

The results are illustrated with three examples: reaction jet control, reaction wheel control, and reaction wheel control with bounded motor torque and speed.

INTRODUCTION

The need for an attitude control system arises frequently in aerospace technology. The Orbiting Astronomical Observatory (see refs. 1 and 2) (OAO) provides a typical example. It consists of a telescope rigidly attached to an Earth-orbiting satellite. The function of the attitude control system of the satellite is to point the telescope in any direction specified by a terrestrial astronomer. His commands may, for example, be step changes in attitude going from one object in the sky to another, or continuous changes which correspond to scanning a portion of the sky or to following a moving object.

The control problem for this satellite falls into two categories. First, while examining a particular object, the satellite is required to hold attitude to an extremely high degree of accuracy. This problem has been studied in several works, such as references 1, 2, and 3. Second, to change from viewing one part of the sky to another, large slewing angles are required. This problem is considerably more difficult from the analytical standpoint than the first one because the describing equations are inherently nonlinear, and any linearization of these equations would be likely to produce meaningless results. Since nonlinear equations must be used, the problem is to find a descriptive means which will allow the determination of the proper control for the slewing motion. Euler angles are often used for the kinematic description of motion. This produces equations that are not only nonlinear but involve complicated combinations of trigonometric terms. The determination of the control in terms of these variables is difficult, and no complete solution to the large angle slew problem exists.

A solution to the slewing problem is proposed in the present report. The solution is complete in the sense that the proposed control laws yield asymptotic stability for all attitudes and attitude changes, and both the kinematic and the dynamic nonlinearities are taken into account. The solution is based on the well-known fact that three-dimensional rotations may be represented by 3×3 orthogonal matrices. Both the system input and the system output are represented by such matrices, and the system error is defined to be the 3×3 orthogonal direction cosine matrix corresponding to the rotation between the actual and desired attitudes of the vehicle. This definition of system error permits the theory of three-dimensional rotations to be applied to the attitude control problem. In particular, Euler's theorem on rotation (see ref. 4) is employed to define a quantitative representation of the system error as the rotation angle of the error matrix. This angle and the magnitude of the angular velocity of the body are used to construct Liapunov functions by means of which stability of the system is investigated. A class of control laws for which the control torque is a function of the real eigenvector of the error matrix and the angular velocity of the controlled body is synthesized. For small errors in attitude these control laws are like those obtained in the Euler angle approach; in addition, they yield asymptotically stable systems for all attitudes.

Several details needed in the main discussion are developed in the appendixes. In particular, notation and special functions together with some of their properties are summarized in appendix A. Some of the consequences of Euler's theorem on rotation are discussed in appendix B. The dynamic equations corresponding to the two methods of generating torque namely by means of reaction jets and reaction wheels are derived in the desired form in appendix C and several aspects of optimal control are considered in appendix D.

LIST OF SYMBOLS

$\bar{A}_{as}$	linear transformation representing the actual attitude of vehicle
A_{as}	matrix representing $\bar{A}_{as}$ with respect to the s basis
$\bar{A}_{ds}$	linear transformation representing the desired attitude of vehicle
A_{ds}	matrix representing $\bar{A}_{ds}$ with respect to the s basis

c	real unit eigencolumn of R corresponding to the eigenvalue +1
$g(A_{as}, w_a)$	nonlinear (gyroscopic) part of the dynamic equation
h_a	representation of the angular momentum $\bar{h}$ in the a basis
h_s	representation of the angular momentum $\bar{h}$ in the s basis
$h_s(0)$	initial angular momentum of the system
h_{max}	maximum angular momentum capacity of a set of reaction wheels
I	the identity matrix
J_a	representation of the moment of inertia operator in the a basis
j_i	eigenvalues of J_a
j_{max}	maximum eigenvalue of J_a
R	error matrix defined by $A_{as}A_{ds}^t$
$\bar{w}_a$	angular velocity of the a basis relative to the s basis
$\bar{w}_d$	angular velocity of the d basis relative to the s basis
w_a	representation of the vector $\bar{w}_a$ in the a basis
w_d	representation of the vector $\bar{w}_d$ in the d basis
z_a	representation of the control torque in the a basis
z_{max}	spherical limit on the control torque
μ_1	measure of mass asymmetry for the reaction jet control
μ_2	measure of mass asymmetry for the reaction wheel control
α	inverse of the moment of inertia of a spherically symmetric mass
φ	error angle
φ_s	point of saturation in the function $\mathrm{sat}(\varphi, \varphi_s)$

Special Symbols

$\dot{y}$	time derivative of the column matrix y
$\dot{Y}$	time derivative of the matrix Y
$\|y\|$	$(y^t y)^{1/2}$
y^t	transpose of the column y
Y^t	transpose of the matrix Y
Y^{-1}	inverse of the matrix Y
$tr(Y)$	trace of the matrix Y (see eq. (A1))
$q(Y)$	column function of the matrix Y (see eq. (A2))
$S(y)$	matrix function of the column y (see eq. (A3))
$sat(\varphi,\varphi_s)$	saturation function = φ/φ_s for $0 \leq \varphi \leq \varphi_s$ and 1 for $\varphi \geq \varphi_s$
$\underset{x \in X}{\operatorname{argmax}} f(x)$	that value of x in X which maximizes $f(x)$
$\underset{x \in X}{\operatorname{argmin}} f(x)$	that value of x in X which minimizes $f(x)$
∇_x	gradient with respect to (x_1,x_2,x_3)

ANALYSIS OF VEHICLE MOTION

The attitude of a rigid body relative to external space can be completely specified by locating a Cartesian set of coordinates fixed in the rigid body (see ref. 4). The orientation of the body set of coordinates relative to any other set with common origin may be described by a matrix of direction cosines. This matrix will be defined as the output of an attitude control system.

Consider two right-hand orthonormal triplets of vectors with the common origin O: $s = (\overline{u}_{s1},\overline{u}_{s2},\overline{u}_{s3})$ and $a = (\overline{u}_{a1},\overline{u}_{a2},\overline{u}_{a3})$, respectively. Let the triplet a be fixed in the body, and let the triplet s be fixed in inertial space. Then, the attitude of the body relative to s will be defined by a transformation which maps s into a. Let such a transformation be denoted by $\overline{A}_{as}$, so that

$$\overline{u}_{ai} = \overline{A}_{as}\overline{u}_{si} \quad , \qquad i = 1,2,3 \tag{1}$$

Let $\overline{A}_{as}$ be represented with respect to the s basis by the 3×3 matrix A_{as}. Since A_{as} is a matrix of direction cosines (i.e., $a_{ij} = \overline{u}_{ai} \cdot \overline{u}_{sj}$) it is orthogonal. Thus,

$$A_{as}A_{as}^{t} = I \tag{2}$$

where A_{as}^{t} is the transpose of A_{as} and I is the identity matrix. Equation (2) indicates that the output of an attitude control system may be represented by an orthogonal matrix; this fact is of primary importance in the present note. The properties of three-dimensional rotation matrices, listed in appendix B, are used in the following development.

Suppose that the triplet a is rotating relative to the triplet s. Then A_{as} is a function of time, say $A_{as}(t)$. Hence, according to equation (2) $A_{as}(t)A_{as}^{t}(t) = I$ for all t, and

$$\dot{A}_{as}A_{as}^{t} + A_{as}\dot{A}_{as}^{t} = \dot{A}_{as}A_{as}^{t} + (\dot{A}_{as}A_{as}^{t})^{t} = 0$$

where the dot indicates time differentiation. Consequently, $\dot{A}_{as}A_{as}^{t}$ must always be a skew-symmetric matrix, say S. From the definition of angular velocity it follows (see appendix B) that if the column matrix w_a represents (with respect to the a triplet) the angular velocity vector $\overline{w}_a$ of the a triplet relative to the s triplet, then (see ref. 5)

$$S = S(w_a) = \begin{pmatrix} 0 & w_{a3} & -w_{a2} \\ -w_{a3} & 0 & w_{a1} \\ w_{a2} & -w_{a1} & 0 \end{pmatrix} \tag{3}$$

Thus, the rotation matrix A_{as} and the column matrix w_a which define the attitude and the angular velocity of the controlled rigid body, respectively, are connected by the matrix differential equation

$$\dot{A}_{as} = S(w_a)A_{as} \quad , \qquad A_{as}(0)A_{as}^{t}(0) = I \tag{4}$$

Equation (4) will be referred to as the matrix form of the kinematic equation. It applies to all attitude control problems.

Unlike the kinematic equation, the dynamic equation depends on the particular method of generating control torque. In the present report, two methods will be discussed. The first is one in which the control torque is external (i.e., control by means of a set of reaction jets). The second is one in which the total angular momentum of the system is conserved, and the control torque is generated internally by a momentum exchange device (i.e., a set of reaction wheels).

The dynamic equations corresponding to these two schemes are derived in appendix C. Both equations are of the following form:

$$\dot{w}_a = J_a^{-1} z_a + g(A_{as}, w_a) \tag{5}$$

where J_a is a constant matrix, measuring moment of inertia; z_a is the column matrix representing the control torque vector with respect to the body axes (the a triplet) and g is a column matrix (representing the gyroscopic acceleration) which is a nonlinear function of the attitude and angular velocity of the controlled rigid body.

When the control torque is external,

$$g = J_a^{-1} S(w_a) J_a w_a \tag{6a}$$

where J_a is the matrix which represents with respect to the body axes the moment of inertia operator of the rigid body being controlled.

When the control is internal, through a set of reaction wheels, and the total system angular momentum is conserved,

$$g = J_a^{-1} S(w_a) A_{as} h_s(0) \tag{6b}$$

where J_a represents, relative to the body axes, the inactive moment of inertia (i.e., the inertia of the rigid body plus locked wheels minus the inertias of the wheels about their spin axes). The column matrix $h_s(0)$ represents, with respect to inertial space, the total angular momentum of the system. It is constant when external torques are absent.

The kinematic equation together with the appropriate dynamic equation describes the controlled object from the point at which the control torque is generated to the output attitude. In order to employ feedback control, the attitude A_{as} of the controlled body must be known. The function of an attitude sensor is to measure A_{as}.

It will be assumed henceforth that the independent controlling variable is the control torque z_a and that A_{as} is measurable directly; that is, any dynamic elements between the applied voltage and the control torque, and between the sensor variables and the corresponding output voltage will be neglected in the sequel. The equations of the plant are then the following:

$$\dot{A}_{as} = S(w_a) A_{as} \tag{7a}$$

$$\dot{w}_a = J_a^{-1} z_a + g(A_{as}, w_a) \tag{7b}$$

Consider, next, the way in which the system error may be defined. Let $\bar{A}_{ds}(t)$ be a transformation given as a function of time which defines the desired attitude of the controlled body (i.e., the input) relative to the s triple (inertial space). The matrix $A_{ds}(t)$ will represent $\bar{A}_{ds}(t)$ with respect to s. The attitude control problem consists in specifying a control law which will force the attitude A_{as} of the controlled body to approximate $A_{ds}(t)$ in some sense as nearly as possible consistent with the existing constraints.

When $A_{as} \neq A_{ds}(t)$ an error exists, and the question arises as to how to specify it. An obvious choice is to define the error by the matrix $E = A_{ds} - A_{as}$ with the desired condition given as $E = 0$. This choice will not be made because the significant property of orthogonality would be lost: E is not orthogonal. A more advantageous choice for a definition of the error is to define it by the orthogonal matrix

$$R = A_{as}A_{ds}^{t}(t) \tag{8}$$

with the desired condition given by $R = I$. The matrices A_{ds}, A_{as}, and R will be referred to as the input, output, and error matrices of the system, respectively.

The preceding discussion is summarized in the form of the block diagram of figure 1.

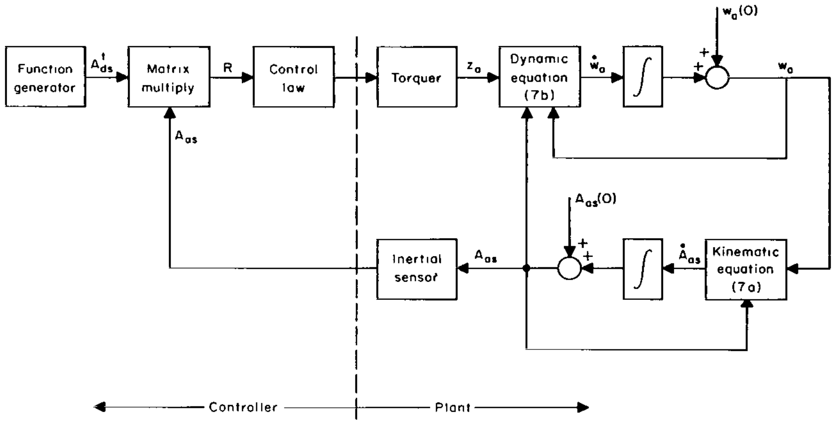


Figure 1.- The block diagram of an attitude control system.

THE SYNTHESIS OF CONTROL LAWS

Let the error matrix R be defined by equation (8). According to Euler's theorem on rotations (see ref. 4), the matrix R may be considered at every instant of time to represent a single rotation. The angle φ of this rotation is given by the following equation (see appendix B).

$$\varphi = \cos^{-1}\left\{\frac{1}{2}\left[\operatorname{tr}(R) - 1\right]\right\}, \qquad 0 \leq \varphi \leq \pi \tag{9}$$

Clearly, $\varphi = 0$ if and only if $R = I$; otherwise, $\varphi > 0$. It is, therefore, natural to consider φ as a quantitative representation of the attitude error. Indeed, a sufficient condition for driving R to I is to maintain the time rate of change of φ negative.

The time derivative of equation (8) is

$$\dot{R} = S(w_a)R - RS(w_d) \tag{10}$$

where $S(w_d)$ depends only on the input. Namely, $S(w_d) = \dot{A}_{ds}A_{ds}^t$. According to the property (A11) of appendix A, $RS(w_d) = S(Rw_d)R$; hence,

$$\mathring{R} = S(w_a - Rw_d)R \tag{11}$$

Equation (11) will be referred to as the kinematic equation of the system error. The column matrix $w_a - Rw_d$ represents the error velocity vector $\bar{w} = \bar{w}_a - \bar{w}_d$ with respect to the body axes.

Let c denote the real unit eigenvector of R. The existence of c is guaranteed by Euler's theorem. According to equation (B7a) of appendix B, which is true for any rotation matrix,

$$\dot{\varphi} = c^t(w_a - Rw_d)$$

But $c^tR = c^t$; hence,

$$\dot{\varphi} = c^t(w_a - w_d) \tag{12}$$

Equation (12) implies that any control which maintains the projection of w_a on c less than the projection of w_d on c throughout the control interval will force the vehicle into the desired attitude.

Kinematic Control

Generally speaking, when the vehicle is controlled by means of an angular momentum exchange device (i.e., reaction wheels or control moment gyros) the angular momentum capacity of the device is small, while the torque levels of the motors, which cause the exchange, are high. Consequently, when φ is far from 0, the time required to perform the exchange is sufficiently short, relative to changes in c, to be practically instantaneous. That is, when φ is far from 0, it may be possible to approximate the attitude control system by a purely kinematic model. In that case the independent controlling variable becomes the angular velocity of the controlled body, and a possible control law is the following:

$$w_a = \underset{w_a \in W_a}{\operatorname{argmin}}\, w_a^t c \tag{13}$$

where W_a is the set of vehicle angular velocities which are consistent with the capacity of the momentum exchange device.

Small Error Control

In the neighborhood of the point $\varphi = 0$, the kinematic model becomes inadequate because of integration lag. However, in this case the behavior of the system may be investigated by means of the following equation, which is equivalent to equation (10) to first-order terms in φ (see appendix B)

$$\dot{q}(R) = S(w_d)q(R) - w_d + w_a \tag{14}$$

where $q(R)$ is a scaled real eigenvector of R, namely, $q(R) = \sin\varphi\, c$. It is given explicitly as a continuous function of the elements of the matrix R in appendix A by equation (A2).

For example, suppose that the vehicle is controlled by means of a set of reaction wheels, that the total angular momentum of the system is zero, and that the reference attitude is constant. Then equation (14) together with the dynamic equations (7b) and (6) with $h_s(o) = 0$ implies that

$$\ddot{q}(R) - J_a^{-1} z_a = 0 \tag{15}$$

In particular, let the control law be the following, where A and B are constant matrices.

$$z_a = -Aq(R) - B\dot{q}(R) \tag{16}$$

Then the behavior of the system near $R = I$ is specified by the following differential equation

$$\ddot{q}(R) + J_a^{-1}B\dot{q}(R) + J_a^{-1}Aq(R) = 0 \tag{17}$$

It may be noted that the above linear differential equation with constant coefficients is independent of the nominal attitude defined by A_{ds}. On the other hand, in the conventional approach (see ref. 3) based on Euler angles the coefficients of the perturbation equation are functions of the Euler angles of A_{ds}. This means that stability (for example) must be investigated for every expected nominal attitude. Such is not the case if equation (17) is used.

Restricted Dynamic Control

Next, suppose that neither a kinematic model nor a perturbation model describes the system adequately. Suppose that the vehicle is described by equation (7), and that it is desired to rotate it from one attitude to another attitude with zero initial and final velocities. A possible scheme for achieving this maneuver is to rotate the vehicle about the eigenvector of the error matrix R until the desired attitude is reached. The equations of motion corresponding to such a control scheme may be derived as follows.

According to (12) when $w_d = 0$,

$$\dot{\varphi} = c^t w_a \tag{18}$$

Since the control rotates the vehicle about the eigenvector c of R, the acceleration $\dot{w}_a$ as well as the velocity w_a must be parallel to c. That is, the acceleration must be of the following form where f is a scalar.

$$\dot{w}_a = -fc \tag{19}$$

Consequently, the error angle φ satisfies the following differential equation:

$$\ddot{\varphi} + f = 0 \tag{20}$$

If f depends on the state (R, w_a) in such a way that it may be expressed as a function of φ and its derivatives only, then (20) is the equation of motion of the system. Thus, if the dynamic equation of the system is given by (7b) and if the control law

$$z_a = -f(\varphi, \dot{\varphi}) J_a c - J_a g(RA_{ds}, w_a) \tag{21}$$

is selected, then the following equation of motion of the system results.

$$\ddot{\varphi} + f(\varphi, \dot{\varphi}) = 0 \tag{22}$$

It should be noted that the control just described is possible only when the initial angular velocity of the vehicle is either zero or an eigenvector of the initial error matrix. If there is velocity orthogonal to the eigenvector, then the direction of the eigenvector cannot be kept constant with finite accelerations.

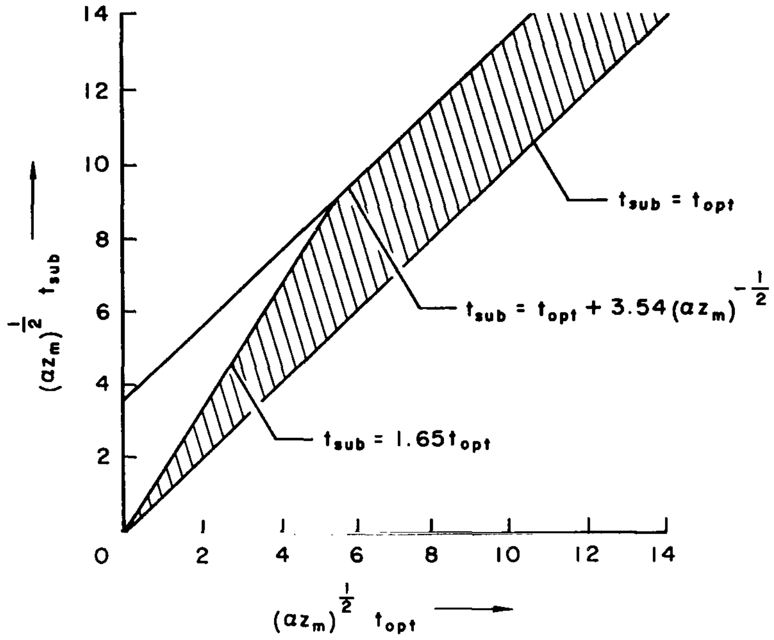
A case for which $R(0)w_a(0) \neq w_a(0)$ may be treated as follows. (It is still assumed that the input is an attitude step.) Select a control law which applies torque so that the vehicle decelerates along $w_a(0)$ until time t_1 when $w_a = 0$; then rotate the vehicle about the real eigenvector of the resulting $R(t_1)$ as discussed above. Such a control is simple to implement and may be attractive in practice if the behavior of the resulting control system does not differ much from that of a specified optimal system. For example, suppose that a system having the dynamic equation,

$$\dot{w}_a = \alpha z_a \tag{23}$$

where α is a constant scalar (i.e., eq. (5) with $J_a = I/\alpha$ and $g = 0$) and $\|z_a\| \leq z_{max}$, is subjected to the following control law:

$$z_a = \left\{ \begin{array}{l} -z_{max} w_a(0)/ \|w_a(0)\| \quad \text{until } w_a = 0\text{, followed by the} \\ \text{time-optimal control about the real eigenvector of} \\ \text{the resulting } R. \end{array} \right\} \tag{24}$$

The behavior of this control law with respect to time-optimality was judged by means of a digital computer simulation. The procedure was to drive the system from the point $(R = I, w_a = 0)$ to an arbitrary point $[R(t_{opt}), w_a(t_{opt})]$ by means of the time-optimal control (see appendix D), and then apply the suboptimal control (24) to bring the resulting point to $[R(t_{opt} + t_{sub}) = I, w_a(t_{opt} + t_{sub}) = 0]$, and compare the two times t_{opt} and t_{sub}. The results are summarized in sketch (a). The data indicate that if a departure in the response time by 65 percent is permitted, the simple suboptimal control is adequate.



Sketch (a).- The shaded region contains 1500 points corresponding to 1500 different initial conditions.

Dynamic Regulator - Asymptotic Stability

The last result to be included in the present report deals with the conditions for which attitude control systems, governed by a class of control laws, are asymptotically stable. It is assumed that the reference attitude A_{ds} is constant for $t > 0$, and that the independent controlling variable is the torque z_a. The pertinent set of differential equations is the following (see eqs. (7b) and (11) with $w_d = 0$).

$$\dot{R} = S(w_a)R \tag{25a}$$

$$\dot{w}_a = J_a^{-1}z_a + g(RA_{ds},w_a) \tag{25b}$$

Let the state space (see ref. 6) for the set (25) be chosen to be

$$X = [(R,w_a) \quad \text{such that} \quad RR^t - I = 0]$$

Suppose that a control law $z_a = f(R,w_a)$ is defined on X. The trajectory of (25) subjected to this law will be denoted by $x(t;x_o)$, where

$$x(0;x_o) = x_o = [R(0), \quad w_a(0)]$$

A region P defined on X and containing the point $(I,0)$ will be called a region of asymptotic stability of (25) subjected to the given control law if for every x_o in P, $x(t;x_o)$ is in P for all $t > 0$ and $x(t;x_o) \to (I,0)$ as $t \to \infty$.

Let the following two scalar functions be defined on X.

$$\varphi = \varphi(R) = \cos^{-1}\left\{\frac{1}{2}\,[\mathrm{tr}(R) - 1]\right\} \tag{26a}$$

$$w = w(w_a) = (w_a^t w_a)^{1/2} \tag{26b}$$

where φ is the angle of R defined by equation (19), and w is the norm of w_a. The image of X under (26) will be denoted by Y. Clearly,

$$Y = [(\varphi,w) \quad \text{such that} \quad 0 \leq \varphi \leq \pi \quad , \quad 0 \leq w]$$

The image under (26) of a trajectory $x(t;x_o)$ will be denoted by $y(t;y_o)$ where

$$y(0;y_o) = y_o = \{\varphi[R(0)], \quad w[w_a(0)]\}$$

In general, through a given point in Y there will pass more than one trajectory. Hence, in general, it is not possible to consider the trajectories $y(t;y_o)$ to be solutions to a differential equation of the form $\dot{y} = F(y)$, and to construct a region of asymptotic stability on Y from the properties of $F(y)$. However, in some cases it is possible to construct a region of asymptotic stability on Y directly from the properties of the trajectories $y(t;y_o)$. This is done next.

Let $Q(m,n)$ be a subset of Y defined by

$$Q(m,n) = [(\varphi,w) \quad \text{such that} \quad 0 \leq \varphi \leq m \leq \pi \quad , \quad 0 \leq w \leq n]$$

and let the scalar function $V(\varphi,w)$ (tentative Liapunov function) be defined on $Q(m,n)$ as follows:

$$V(\varphi,w) = \int_0^{\varphi} g_1(s)ds + \int_0^{w} [s/g_2(s)]ds \tag{27}$$

where the scalar functions $g_1(s)$ and $g_2(s)$ are such that for $m,n > 0$ and $M,N < \infty$,

$$0 < g_1(s) < M \quad \text{for} \quad 0 < s \leq m, \quad g_1(0) < M \tag{28a}$$

$$0 < g_2(s) < N \quad \text{for} \quad 0 \leq s \leq n \tag{28b}$$

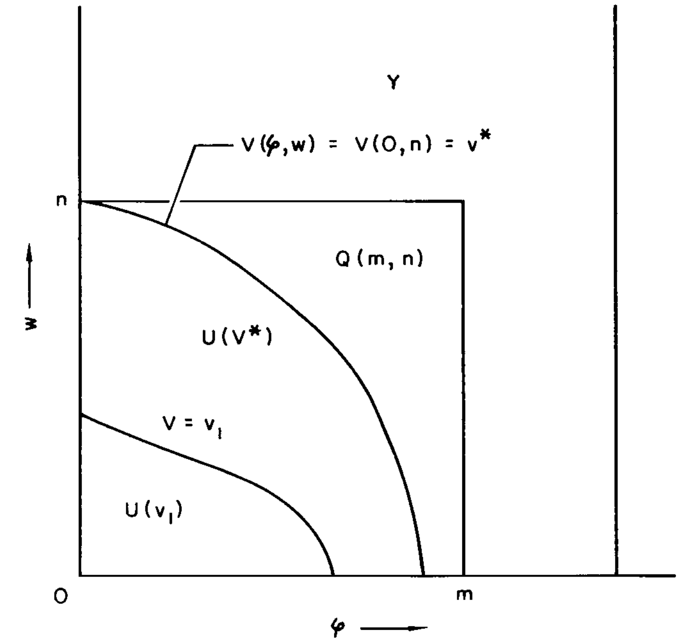
Then $V(0,0) = 0$, and $V(\varphi,w) > 0$ for other points in $Q(m,n)$. Moreover, the contours $V(\varphi,w) = v_1$ and $V(\varphi,w) = v_2$ have no points in common if $v_1 \neq v_2$. Finally, $V(\varphi_2,0) > V(\varphi_1,0)$ if, and only if, $\varphi_2 > \varphi_1$; similarly, $V(0,w_2) > V(0,w_1)$ if, and only if $w_2 > w_1$. Therefore, the sectors in Q defined by

$$U(v) = \left\{(\varphi,w) \quad \begin{matrix} \text{such that} \quad V(\varphi,w) < v < v^* = \min\,[V(m,0),\, V(0,n)] \\ \text{and} \quad \varphi \geq 0,\ w \geq 0 \end{matrix}\right\} \tag{29}$$

are nested by v, namely,

$$U(v_2) \subset U(v_1) \qquad \text{if} \qquad 0 \leq v_2 < v_1 \leq v^*$$

(See sketch (b)).



Sketch (b)

Hence, if there exists a number $b \leq v^*$ such that for any trajectory $y(t;y_o)$, with y_o in $U(b)$, the time derivative of v, along that trajectory, is negative semidefinite, namely,

$$\dot{v} \leq 0$$

and if $\dot{v} = 0$ for at most a countable number of points along the trajectory, then (see ref. 7) the region $U(b)$ is a region of asymptotic stability on Y; while, the region $P(b)$ of points (R,w_a) in X which maps under (26) into $U(b)$ is a region of asymptotic stability of (25) subjected to the given control law.

Suppose that the control law has the following form:

$$z_a = -J_a[g_1(\varphi)g_2(w)c + g_3(\varphi,w)w_a] \tag{30}$$

where $g_1(\varphi)$ and $g_2(w)$ satisfy conditions (28), c is the eigenvector of R, and $g_3(\varphi,w)$ satisfies the relation

$$w^2 g_3(\varphi,w) \geq w_a^t g(RA_{ds},w_a) \tag{31}$$

on a sector $U(b)$ where $b > 0$; then

$$P(b) = [(R,w_a) \quad \text{such that} \quad (\varphi,w) \quad \text{is in} \quad U(b)]$$

is a region of asymptotic stability of (25) subjected to the control law (30).

Indeed,

$$\dot{v} = \frac{\partial}{\partial\varphi} V(\varphi,w)\dot{\varphi} + \frac{\partial}{\partial w} V(\varphi,w)\dot{w}$$

which on using (25b) and (27), the fact that $\varphi = w_a^t c$ (see eq. (12) with $w_d = 0$) and the control law (30), becomes

$$\dot{v} = g_2^{-1}[w_a^t g(RA_{ds},w_a) - w^2 g_3(\varphi,w)]$$

Hence, if condition (31) holds,

$$\dot{v} \leq 0$$

Moreover, from (30), (25b), and (6) (which imply $g = 0$ if $w = 0$) it is clear that $\dot{w}_a \neq 0$ for $w = 0$ and $\varphi \neq 0$. Hence, $\dot{v} = 0$ for, at most, a countable number of points along the trajectory. Consequently, $P(b)$ is a region of asymptotic stability of (25) subjected to the control law (30) with condition (31).

Two special cases of (25) will be used to illustrate the application of the above-derived result. The first case is one in which $g(RA_{ds},w_a)$ in equation (25b) is defined by (6a) and corresponds to control by means of a set of reaction jets. The second case is one in which $g(RA_{ds},w_a)$ is defined by (6b) and corresponds to control by means of a set of reaction wheels.

<u>Case I</u>: Let the nonlinear part g of the dynamic equation (25b) be defined by (6a), and consider the special case of control law (30) with $g_1(\varphi) = k_1\varphi, g_2(w) = 1$, $g_3(\varphi,w) = k_2$, where k_1 and k_2 are constant positive scalars. Then (30) becomes essentially a proportional plus rate control:

$$z_a = -J_a(k_1\varphi c + k_2 w_a) \tag{32}$$

Clearly, conditions (28) are satisfied everywhere on Y. According to equation (27),

$$V(\varphi,w) = \frac{1}{2} k_1\varphi^2 + \frac{1}{2} w^2$$

Moreover, as is shown in appendix C, where the scalar μ_1 is defined (C3),

$$w_a^t g(RA_{ds},w_a) = w_a^t J_a^{-1} S(w_a) J_a w_a = \mu_1 w_{a_1} w_{a_2} w_{a_3} \leq 3^{-3/2} \mu_1 w^3$$

Therefore, condition (31) is satisfied on U(b) where $b = (27/2)(k_2/\mu_1)^2$. That is, the system is asymptotically stable on the following region of X.

$$\frac{1}{2} k_1 \left\{\cos^{-1}\left[\frac{1}{2} tr(R) - \frac{1}{2}\right]\right\}^2 + \frac{1}{2} w_a^t w_a \leq \left(\frac{27}{2}\right)\left(\frac{k_2}{\mu_1}\right)^2$$

It may be noted that when the vehicle has a spherical mass distribution, which implies that $\mu_1 = 0$, the system is asymptotically stable everywhere on X.

Case II: Next suppose that control (32) is applied to a reaction wheel control system. The nonlinear part of the dynamic equation is defined by (6b). It is shown in appendix C, where the constant scalar μ_2 is defined (eq. C7), that

$$w_a^t g(RA_{ds},w_a) = w_a^t J_a^{-1} S(w_a) RA_{ds} h_s(0) \leq \mu_2 w^2 [h_s^t(0) h_s(0)]^{1/2}$$

Hence, the system is asymptotically stable everywhere on X if

$$k_2 > \mu_2 \, \|h_s(0)\|$$

Thus, in both cases the simple "proportional plus rate" control law (32) may be used for the synthesis of asymptotically stable attitude control systems.

Usually, a control system must not only be stable, but also be reasonably stiff to disturbances and must respond quickly to commands; it must not require excessive torques, velocities, power, etc. The freedom which remains in the selection of the functions g_i in (30) after conditions (28) and (31) are satisfied may be used by a designer to consider such additional qualities of the system.

A control law for the OAO.- Consider, for example, the OAO. It is controlled by means of a set of reaction wheels, but unlike case II discussed above, both the control torque z_a and the wheel speeds must not exceed some preassigned values, and a fast responding system is desired. A control law which may be adequate in such a case is given below:

$$z_a = -(1/2)\left(z_{max}/j_{max}\right)J_a\left[(1 - k)sat(\varphi,\varphi_s)c + (1 + k)w_a/w_{max}\right] \qquad (33)$$

where j_{max} is the maximum eigenvalue of the inertia matrix J_a and

$$k = \mu_2 j_{max} \|h_s(0)\| w_{max}/z_{max} < 1$$

$$\varphi_s = 2(1 - k)(1+k)^{-2}(j_{max}w_{max}^2/z_{max})$$

$$\begin{aligned} sat(\varphi,\varphi_s) &= \varphi_s^{-1}\varphi \qquad \text{for} \quad 0 \le \varphi \le \varphi_s \\ &= 1 \qquad\quad \text{for} \quad \varphi \ge \varphi_s \end{aligned}$$

Control law (33) is a special case of the control law (30). Thus, (30) becomes (33) if the following identifications are made:

$$g_1 = sat(\varphi,\varphi_s) \ , \quad g_2 = 1/2\ (1 - k)z_{max}/j_{max} \ , \quad g_3 = 1/2\ (1 + k)z_{max}/(j_{max}w_{max})$$

Equations (33), (25b), and (6b) imply that $\dot{w} < 0$ for all R and $w \ge w_{max}$. Therefore, the trajectory of the system cannot leave the region

$$T = [(R,w_a) \quad \text{such that} \quad (w_a^t w_a)^{1/2} \le w_{max}]$$

Equation (33) implies that $(z_a^t z_a)^{1/2} \le z_{max}$ on T.

Finally, according to the discussion of case II, the system is asymptotically stable everywhere on X if

$$1/2(1 + k)z_{max}/(j_{max}w_{max}) > \mu_2 \|h_s(0)\|$$

which is true in the present case. Therefore, the system has the following properties:

i) It is asymptotically stable on T.

ii) The torque is bounded by z_{max} on T.

iii) If $w_{max} = [h_{max} - \|h_s(0)\|]/j_{max}$ where h_{max} is a spherical bound on the momentum capacity of the reaction wheels, then the wheels will not saturate on T.

iv) In the neighborhood of point $(I,0)$ the system behaves as a three-dimensional second-order linear system with damping of 0.5 and natural frequency of $0.5(1+k)z_{max}(j_{max}w_{max})^{-1}$.

v) The plots in figures 2, 3, and 4 depict some of the results obtained from a digital computer simulation of an OAO-type vehicle. The inertia matrix J_a was assumed to be diagonal, $J_a = (999,1110,1410)$ kg-m^2; the saturation limits for the momentum exchange and control torque were assumed to be $h_{max} = 4.68$ N-m-sec and $z_{max} = 0.231$ N-m, respectively. (See ref. 2.)

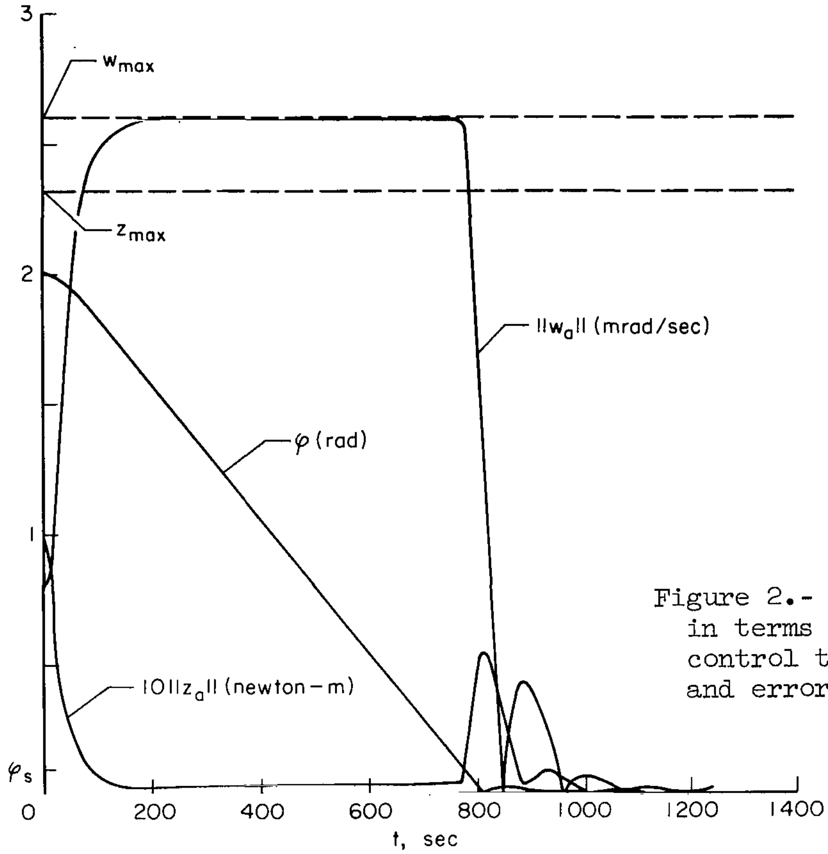


Figure 2.- The response of the system in terms of the magnitudes of the control torque, angular velocity, and error angle.

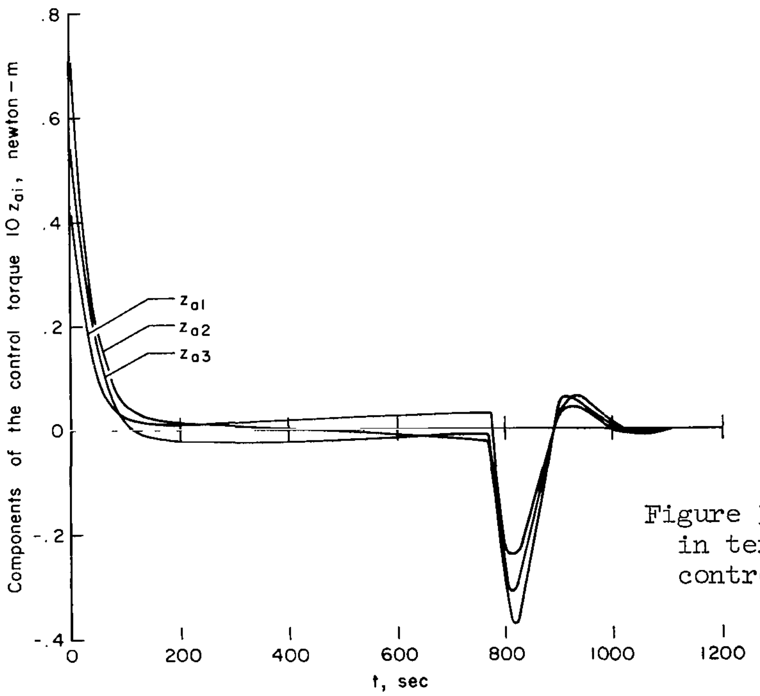


Figure 3.- The response of the system in terms of the components of the control torque.

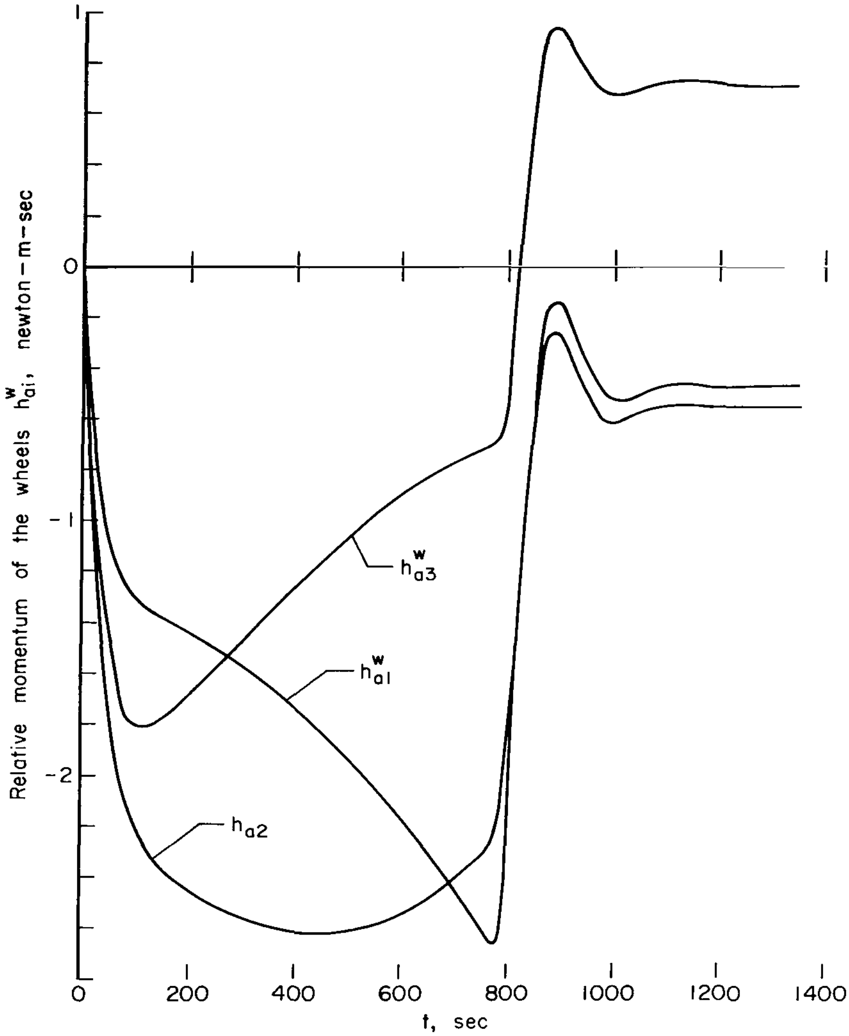


Figure 4.- The response of the system in terms of the components of the relative momentum of the reaction wheels.

At $t = 0$ $\varphi = 2$ rad, $c_a^t = -3^{-1/2}(1,1,1)$, $w_a^t = 0.5(1,-1,-1)$ mrad/sec. The wheels were assumed to be locked prior to $t = 0$; $w_{max} = 2.6$ mrad/sec.

The control may be divided roughly into three parts. For $0 \leq t \leq 100$ sec, the control generates a pulse-like torque to bring the vehicle to its maximum velocity. For $100 \leq t \leq 750$ sec the vehicle coasts with maximum velocity; the small torques probably counteract the gyroscopic effects. For $t \geq 750$ sec, the control again generates pulse-like torques to stop the vehicle on target. After the transient, the initial momentum of the vehicle resides in the sheels (fig. 4). One may suppose that although the duration of these intervals may change with initial conditions, the general shape of the response does not.

CONCLUDING REMARKS

The representation of input and output of an attitude control system by 3×3 orthogonal matrices led to the definition of the system error as the 3×3 orthogonal matrix (the error matrix) corresponding to the rotation (the error rotation) between the actual attitude of the controlled body and the desired attitude. Euler's theorem on rotations was employed to express the error matrix in terms of four parameters, namely, the angle (the error angle) and the direction cosines of the real eigenvector of the error matrix. It was discovered that the error angle and the magnitude of the angular velocity of the controlled body are convenient variables in the construction of Liapunov functions for the process. A class of control laws was synthesized for which the control torque is a function of the real eigenvector of the error matrix and the angular velocity of the controlled body. Conditions for which any member of the class yields an asymptotically stable control system were stated.

The results presented are not applicable to on-off control. If they are to be applied to a reaction jet control system, throttling or pulse width modulation schemes must be employed. The results are most applicable to reaction wheel control systems. It appears that they may be made to apply to other momentum exchange control schemes (i.e., control moment gyros).

The errors and delays in the attitude sensors, computer, and motors were neglected in the analysis. To estimate the significance of such effects physical sensors and motors were placed on an air bearing platform, and the loop was closed with a real time digital computer. The simulation of the OAO-type vehicle was repeated and no significant departures from theory (simple computer simulation) were observed.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., July 1, 1966
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APPENDIX A

NOTATION AND SPECIAL FUNCTIONS

In this report vectors are denoted by lower-case letters with an overbar. Linear transformations are denoted by upper-case letters with an overbar. The term triplet (or basis) always means a right-hand triplet of orthonormal vectors. Several triplets are employed; a letter is associated with each basis to distinguish one from another. All triplets are erected at the fixed point of the controlled body. A basis vector is denoted by the letter "u" with an overbar and two subscripts; the first subscript indicates the basis to which the vector belongs, and the second subscript indicates the place the vector occupi occupies in the triplet. For example, the symbol $\overline{u}_{a2}$ stands for the second vector of the a triplet (or a basis).

Column matrices representing a vector with respect to the various bases are denoted by the lower-case letter used to denote the vector, but without the overbar and with a subscript which indicates the basis. For example, y_a stands for the column matrix representing the vector $\overline{y}$ with respect to the a basis. All columns are 3 by 1.

A linear transformation which physically corresponds to a change of bases is referred to as a rotation,and a linear transformation which physically corresponds to a linear operation on a vector is referred to as a linear operator. A rotation is denoted by a capital letter with an overbar and two subscripts. The first subscript indicates the image of the basis denoted by the second subscript. For example, $\overline{A}_{as}$ is a rotation such that $\overline{u}_{ai} = \overline{A}_{as}\overline{u}_{si}$, $i = 1, 2, 3$. The matrix representing a rotation such as $\overline{A}_{as}$ will be denoted by A_{as}.

A linear operator is denoted without any subscripts but with an overbar. For example, in the expression $\overline{h} = \overline{J}\,\overline{w}$, $\overline{J}$ is the linear operator which takes the vector $\overline{w}$ into the vector $\overline{h}$. The matrix representing a linear operator with respect to a basis is denoted by the letter denoting the operator, but unbarred and with a subscript which indicates the basis in which the representation is being made. For example, J_a is the matrix representing the operator $\overline{J}$ with respect to the a basis.

The transpose and inverse of a matrix A is denoted by A^t and A^{-1}, respectively. The identity matrix is denoted by I.

Let $A = (a_{ij})$ be an arbitrary 3×3 matrix, and let $y = (y_i)$ be an arbitrary 3×1 column matrix. Then, the functions of the matrix elements are:

The trace of A,

$$\operatorname{tr}(A) = \sum_{i=1}^{3} a_{ii} \tag{A1}$$

The column function of A,

$$q(A) = \frac{1}{2}\begin{pmatrix} a_{23} & -a_{32} \\ a_{31} & -a_{13} \\ a_{12} & -a_{21} \end{pmatrix} \tag{A2}$$

The matrix function of y

$$S(y) = \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \tag{A3}$$

Let $\bar{a}$ and $\bar{b}$ be two arbitrary vectors. The two binary operations $\bar{a} \cdot \bar{b}$ and $\bar{c} = \bar{a} \times \bar{b}$ have the following meaning with respect to an arbitrary (but orthonormal) basis, say the e basis:

$$\bar{a} \cdot \bar{b} = a_e^t b_e \tag{A4}$$

$$\bar{c} = \bar{a} \times \bar{b} \longleftrightarrow c_e = -S(a_e)b_e \tag{A5}$$

The following identities are used in the report. (For arbitrary columns matrices y and z and matrix A)

$$S(y)z = -S(z)y \tag{A6}$$

$$q[S(y)] = y \tag{A7}$$

$$tr[S(y)A] = -2y^t q(A) \tag{A8}$$

Let B be an orthogonal matrix; then

$$tr(BAB^t) = tr(A) \tag{A9}$$

$$q(BAB^t) = Bq(A) \tag{A10}$$

$$S(By) = BS(y)B^t \tag{A11}$$

Let c be a column such that $c^t c = 1$; then

$$S^2(c) = -I + cc^t \tag{A12}$$

APPENDIX B

SOME PROPERTIES OF ROTATIONS

THE MATRIX FORM OF THE KINEMATIC EQUATION

Let the a basis be rotating with angular velocity $\bar{w}$ relative to the s basis, and let the matrix A_{as} represent the rotation between the s basis and the a basis. Let P be an arbitrary point fixed in the a basis. The position of P will be denoted by the vector $\bar{r}$; its velocity relative to the s basis will be denoted by the vector $\bar{v}$. Then, $r_a = A_{as}r_s$, $v_a = A_{as}v_s$, $\dot{r}_s = v_s$, $\dot{r}_a = 0$; and $v_a = -S(w_a)r_a$. Consequently, the following chain of equations is true.

$$0 = \dot{r}_a = \dot{A}_{as}r_s + A_{as}\dot{r}_s = \dot{A}_{as}A_{as}^t r_a + A_{as}v_s$$

$$= \dot{A}_{as}A_{as}^t r_a + v_a = [\dot{A}_{as}A_{as}^t - S(w_a)]r_a$$

Since the chain must be true for every point fixed to the a basis, it follows that the matrix form of the kinematic equation is

$$\dot{A}_{as} = S(w_a)A_{as} \tag{B1}$$

THE (φ, c) PARAMETERS

Consider any rotation matrix A_{as}. According to Euler's theorem on rotations, A_{as} always has the eigenvalue +1. Hence, the transformation $x \to A_{as}x$ may be thought of as a rigid rotation about the direction c ($c^tc = 1$) of the eigenvector of A_{as} corresponding to the eigenvalue +1, through some angle φ. Therefore, the matrix A_{as} may be considered to be the solution at $\tau = \varphi$ of the following differential equation.

$$\frac{d}{d\tau}(A_{as}) = S(c)A_{as}$$

with initial condition, $A_{as} = I$ at $\tau = 0$; that is,

$$A_{as} = \exp[\varphi S(c)] = I + \sin\varphi S(c) + (1 - \cos\varphi)S^2(c) \tag{B2}$$

Property (A12) was used in the representation of the exponential. Thus, equation (B2) defines the rotation matrix A_{as} in terms of the four parameters (φ, c).

Conversely, the (φ, c) parameters of a rotation matrix A_{as} may be determined from the elements of the matrix A_{as} as follows. (See eq. (A2) for the definition of the function $q(A)$.)

$$\varphi = \cos^{-1}\left\{\frac{1}{2}\,[\mathrm{tr}(R) - 1]\right\}, \qquad 0 \leq \varphi \leq \pi \tag{B3a}$$

$$c = \csc\varphi\; q(A_{as}) \qquad\qquad 0 < \varphi < \pi \tag{B3b}$$

The singular cases must be considered separately. When $\varphi = \pi$ the components of c are the solutions of the following set of equations.

$$|c_i| = \left[\frac{1}{2}\,(a_{ii} + 1)\right]^{1/2}, \qquad c_i c_j = \frac{1}{2}\,a_{ij}$$

while, for $\varphi = 0$, c is arbitrary.

The four parameters (φ,c) are constrained by $c^t c = 1$.

THE q PARAMETER

When the rotation angle of A_{as} is known to be restricted to the interval, $0 \leq \varphi(1/2)\pi$, then the matrix A_{as} may be defined by its skew-symmetric part only, as follows. Let (see definition (A2)) $q = q(A_{as})$. Then by equation (B3b)

$$q = \sin\varphi\; c\;, \qquad q^t q = \sin^2\varphi \tag{B4}$$

Equations (B2) and (B4) imply that for $0 \leq \varphi \leq (1/2)\pi$

$$A_{as} = I + S(q) + \left[1 + (1 - q^t q)^{1/2}\right]^{-1} S^2(q) \tag{B5}$$

The components of q constitute a set of independent coordinates of A_{as} for $0 \leq \varphi \leq (1/2)\pi$. Note that $q(A_{as})$ is a continuous function of the elements of the matrix A_{as} for $0 \leq \varphi < (1/2)\pi$.

THE KINEMATIC EQUATION OF THE (φ,c) PARAMETERS

Let the matrix A_{as} be defined in terms of the (φ,c) parameters as in equation (B2). Since c is an eigenvector of A_{as},

$$A_{as}c = c$$

and so,

$$\dot{A}_{as}c + A_{as}\dot{c} = \dot{c}$$

That is,

$$S(w_a)c = (I - A_{as})\dot{c} = -\sin\varphi\, S(c)\dot{c} - (1 - \cos\varphi)S^2(c)\dot{c}$$

But, according to (A12) $S^2(c) = -I + cc^t$, while $c^t\dot{c} = 0$; hence,

$$S(w_a)c = -\sin\varphi\, S(c)\dot{c} + (1 - \cos\varphi)\dot{c}$$

and

$$S(c)S(w_a)c = (1 - \cos\varphi)S(c)\dot{c} + \sin\varphi\,\dot{c}$$

But, for any three columns x, y, and z, $S(x)S(y)z = (x^tz)y - (x^ty)z$. (This is the matrix form of the vector triple product identity.) Hence,

$$w_a = \dot{\varphi}c + \sin\varphi\dot{c} + (1 - \cos\varphi)S(c)\dot{c} \tag{B6}$$

while the kinematic equation is

$$\dot{\varphi} = w_a^tc = c^tw_a \tag{B7a}$$

$$\dot{c} = \frac{1}{2}S(w_a)c + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)[w_a - (w_a^tc)c] \tag{B7b}$$

THE KINEMATIC EQUATION OF THE q PARAMETER

If A_{as} is defined in terms of its skew-symmetric part as in equation (B3), then for $0 \leq \varphi < \pi$,

$$w_a = \dot{q} + f_1(q,\dot{q}), \qquad \dot{q} = w_a + f_2(q,w_a) \tag{B8}$$

where,

$$f_1(q,\dot{q}) = \left[1 - q^2 + (1 - q^2)^{1/2}\right]^{-1}(q^t\dot{q})q + \left[1 + (1 - q^2)^{1/2}\right]^{-1}S(q)\dot{q}$$

and

$$f_2(q,w_a) = \frac{1}{2}S(w_a)q - \frac{1}{2}\left[1 - (1 - q^2)^{1/2}\right]w_a - \frac{1}{2}\left[1 + (1 - q^2)^{1/2}\right]^{-1}(w_a^tq)q$$

It may be noted that both f_1 and f_2 are of order higher than one in $\|q\|$ for $\|q\| = 0$, so that the linear part of (B8) is

$$w_a = \dot{q}, \qquad \dot{q} = w_a \tag{B9}$$

APPENDIX C

DERIVATION OF THE DYNAMIC EQUATION

The equations relating the angular acceleration of the controlled rigid body to the controlling torque will be derived in this appendix. Two cases will be considered. In the first the controlling torque is external (i.e., control by means of a set of reaction jets). In the second the total angular momentum of the system is conserved, and the controlling torque is generated internally by a momentum exchange device (i.e., a set of reaction wheels).

CONTROL BY MEANS OF EXTERNAL TORQUE

Let the two vectors $\bar{h}$ and $\bar{w}$ be the angular momentum and angular velocity, respectively, of the vehicle with respect to inertial space. Let $\bar{J}$ be the moment of inertia operator of the vehicle. By definition,

$$\bar{h} = \bar{J}\,\bar{w}$$

The above equation has the following representation in the s basis (inertial) space) and the a basis (the vehicle), respectively.

$$h_s = J_s w_s$$

$$h_a = J_a w_a$$

The matrix J_a representing the moment of inertia operator of the vehicle in the body coordinates will be constant if, as will be assumed henceforth, the mass distribution of the vehicle is fixed.

If the torque acting on the vehicle is denoted by $\bar{z}$, it follows from Newton's law that

$$\dot{h}_s = z_s$$

But $h_a = A_{as}h_s$; hence,

$$\dot{h}_a = J_a\dot{w}_a = \dot{A}_{as}h_s + A_{as}\dot{h}_s$$

The above equation together with the kinematic equation (C2) implies the following dynamic equation (Euler's equations of motion).

$$\dot{w}_a = J_a^{-1}z_a + J_a^{-1}S(w_a)J_a w_a \qquad \text{(C1)}$$

The following inequality is useful for estimating the effectiveness of the nonlinear part of (C1)

$$\left[\frac{d}{dt}\,\|w_a\|\right]_{z_a = 0} \leq 3^{-3/2}\mu_1\,\|w_a\|^2 \tag{C2}$$

where

$$\mu_1 = j_1^{-1}(j_2 - j_3) + j_2^{-1}(j_3 - j_1) + j_3^{-1}(j_1 - j_2) \tag{C3}$$

and j_i are the eigenvalues of J_a.

The inequality (C2) follows from the fact that $(d/dt)\|w_a\| = w_a^t\dot{w}_a/\|w\|$, and that the maximum of $|w_a^t J_a^{-1} S(w_a) J_a w_a| = \mu_1\,|w_{a1}w_{a2}w_{a3}|$ on the sphere $w_a^t w_a = \|w_a\|^2$ is $3^{-2/3}\mu_1\,\|w_a\|^3$.

CONTROL BY MEANS OF INTERNAL TORQUES

Consider three wheels oriented so that the spin axis of the ith wheel is parallel to the ith vector of the a basis (the vehicle). Let the moment of inertia of the ith wheel about its spin axis be denoted by j_i^W. Let the matrix J_a^V represent the moment of inertia of the vehicle plus locked wheels with respect to the body axes, and let J_a^W be a diagonal matrix whose elements are the moments of inertia j_i^W of the wheels about their spin axes. Then the total angular momentum $\bar{h}$ of the system may be represented in the a basis as follows:

$$h_a = J_a^V w_a + J_a^W w_a^W$$

where the column matrix w_a^W has the elements w_{ai}^W which are the spin velocities of the wheels relative to the vehicle. But $h_a = A_{as}h_s$. Hence,

$$J_a^V w_a + J_a^W w_a^W = A_{as}h_s \tag{C4}$$

Taking the time-derivative of equation (C4),

$$J_a^V \dot{w}_a + J_a^W \dot{w}_a^W = A_{as}h_s + A_{as}\dot{h}_s$$

Let the motor torque acting on the ith wheel be denoted by $-z_{ai}$. Then,

$$J_a^W(\dot{w}_a^W + \dot{w}_a) = -z_a$$

where the column matrix z_a has the elements z_{ai}. Hence,

$$\dot{w}_a = J_a^{-1}z_a + J_a^{-1}S(w_a)A_{as}h_s + J_a^{-1}A_{as}\dot{h}_s$$

where, J_a is the inertia of the vehicle plus locked wheels minus the inertias of the wheels about their spin axes; namely,

$$J_a = J_a^V - J_a^W$$

If there is no external torque, the total angular momentum of the system is conserved, in which case the dynamic equation is the following:

$$\dot{w}_a = J_a^{-1} z_a + J_a^{-1} S(w_a) A_{as} h_s \;, \qquad h_s = \text{constant} \tag{C5}$$

The following inequality is useful for estimating the effectiveness of the nonlinear part of (C5)

$$\left[\frac{d}{dt}\,\|w_a\|\right]_{z_a = 0} \leq \mu_2 \,\|w_a\|\,\|h_s\| \tag{C6}$$

where

$$\mu_2 = |j_1^{-1} - j_2^{-1}| + |j_2^{-1} - j_3^{-1}| + |j_1^{-1} - j_3^{-1}| \tag{C7}$$

and j_i are the principal inertias of J_a.

The inequality (C6) follows from the Schwartz inequality and the fact that $\|w_a J_a^{-1} S(w_a)\| \leq \mu_2 \|w_a\|^2$ and $\|A_{as} h_s\| = \|h_s\|$.

APPENDIX D

FORM OF THE TIME OPTIMAL CONTROL LAW

Given the following system of differential equations,

$$\dot{R} = S(w_a)R \tag{D1}$$

$$\dot{w}_a = \alpha z_a \tag{D2}$$

with the controlling variable z_a restricted to the closed sphere $\|z_a\| \leq z_{max}$. The time optimal control maximizes the Hamiltonian (see ref. 8)

$$H = tr(P^t\dot{R}) + y_a^t\dot{w}_a \tag{D3}$$

where the matrix P satisfies the differential equation

$$\dot{P} = S(w_a)P \tag{D4}$$

and the column matrix y_a satisfies the differential equation

$$\dot{y}_a = - \nabla_{w_a}H \tag{D5}$$

Equations (D1) and (D4) have the same transition matrix. Hence,

$$P = RR^t(0)P(0)$$

Consequently,

$$P^t\dot{R} = P^t(0)R(0)R^tS(w_a)R = P^t(0)R(0)S(R^tw_a)$$

Hence, on taking the trace of the above equation and using equation (A8), one obtains the following result.

$$tr(P^t\dot{R}) = - w_aRk$$

where $k = 2q[P^t(0)R(0)]$.

Therefore, the time optimal control has the following form

$$z_a = z_{max}y_a/(y_a^ty_a)^{1/2}$$

where the column y_a is the solution of the following boundary value problem.

$$\dot{R} = S(w_a)R$$

$$\dot{w}_a = \alpha z_{max} y_a / (y_a^t y_a)^{1/2}$$

$$\dot{y}_a = - Rk$$

$[R(0),\ w_a(0)]$, and $(I,0)$ define the boundary values.

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